Conditions for a Vector Subspace $E(t)$ and for a Projector $P(t)$ Not to Depend on t

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ABSTRACT

A projector $P(t)$ may depend on t when its range $E(t)$ does not. Conditions for a vector subspace $E(t)$ and for a projector $P(t)$ not to depend on t are established. Useful models of applications are mentioned.

INTRODUCTION

Conditions for a vector subspace $E(t)$ of a vector space $\mathbb{F}$ not to depend on $t \in \Omega \subset \mathbb{E}$ turn out to have many useful applications. As examples, see the proofs of Lemma 3.4 of [3] and of Theorems 3.5 and 5.3 of [2].

Such a condition appears in the proof of Lemma 3.4 of [3] without being stated, in the particular case where $\mathbb{F}$ is a finite-dimensional normed space and $\mathbb{E} = \mathbb{R}$. This condition is generalized by a quite different proof to the case where $\mathbb{E}$ and $\mathbb{F}$ are any normed spaces and $E(t)$ is finite-dimensional in Theorem 2.2 of [2]. In Corollary 2.3 of [2] the particular case $\mathbb{E} = \mathbb{R}^q$ is considered, and there

$$E(t) = \operatorname{span}\{e_1(t),\dots,e_m(t)\}, \qquad t = (t_1,\dots,t_q) \in \Omega.$$

In this corollary, the main requirement for $E(t)$ be constant is that the partial derivatives $\partial e_i(t)/\partial t_j$ belong to $E(t)$. It may be observed there that these derivatives are not supposed to be continuous.

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Theorem 3 of this paper generalizes Theorem 2.2 of [2] to the case where $E(t)$ is of any dimension. In this theorem

$$E(t) = \pi(t)(\mathbb{E}_0),$$

and in the particular case where $\mathbb{E} = \mathbb{R}$, the essential assumption is that

$$\pi'(t)(\mathbb{E}_0) \subset E(t),$$

where $\pi'(t)$ denotes the derivative of $\pi(t)$.

In Corollary 5, we show that Theorem 2.2 of [2] is an immediate consequence of Theorem 3. In Corollary 4, we consider the case where $E(t)$ is the range of a projector $P(t)$, and then the main assumption is that $E(t)$ is invariant under the derivative of $P(t)$.

This idea leads to Theorem 6, where we show that a projector $P(t)$ is constant if (and obviously only if) it commutes with its derivative.

Lemma 2, found by the referee [together with Lemma 1(b)], permitted a very important simplification of the proof of the main theorem of this paper, i.e. Theorem 3. It also permitted the generalization, due to the referee, of Corollary 7 by Theorem 6.

1. PRELIMINARIES

NOTATION. $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{H}$ are normed spaces on $\mathbb{K}$. Ω is a nonempty open subset of $\mathbb{E} \cdot C(\Omega, \mathbb{F})$ denotes the set of continuous maps from Ω into $\mathbb{F}$, $\mathscr{L}(\mathbb{E}, \mathbb{F})$ denotes the subset of $C(\mathbb{E}, \mathbb{F})$ of linear maps from $\mathbb{E}$ into $\mathbb{F}$, and $\mathscr{L}(\mathbb{E}) = \mathscr{L}(\mathbb{E}, \mathbb{E})$. $C_s(\Omega, \mathscr{L}(\mathbb{F}, \mathbb{G}))$ denotes the set of *strongly continuous* maps from Ω into $\mathscr{L}(\mathbb{F}, \mathbb{G})$ [T is such a map if, for every $y \in \mathbb{F}$, the map $t \to T(t)y$ belongs to $C(\Omega, \mathbb{G})$]. $D(\Omega, \mathbb{F})$ denotes the subset of $C(\Omega, \mathbb{F})$ consisting of maps which possess a Fréchet derivative at every point of Ω. If $f \in D(\Omega, \mathbb{F})$ and $t \in \Omega$, then $Df(t) \in \mathscr{L}(\mathbb{E}, \mathbb{F})$ denotes the Fréchet derivative of f at the point t. $D_s(\Omega, \mathscr{L}(\mathbb{F}, \mathbb{G}))$ denotes the subset of $C_s(\Omega, \mathscr{L}(\mathbb{F}, \mathbb{G}))$ consisting of *strongly differentiable maps* [$T \in C_s(\Omega, \mathscr{L}(\mathbb{F}, \mathbb{G}))$ is such a map if, for every $y \in \mathbb{F}$, the map $t \to T(t)y$ belongs to $D(\Omega, \mathbb{G})$].

$$D_s \cap C(\Omega, \mathscr{L}(\mathbb{F}, \mathbb{G}))$$

denotes

$$D_s(\Omega, \mathscr{L}(\mathbb{F}, \mathbb{G})) \cap C(\Omega, \mathscr{L}(\mathbb{F}, \mathbb{G})).$$

If T is a map from Ω into $\mathscr{L}(\mathbb{F},\mathbb{G})$, $y\in\mathbb{F}$, $t\in\Omega$, and if the map $s\to T(s)y$ possesses a Fréchet derivative at the point $s=t$, then this derivative is denoted by $DT(t)y$ or by $DT(s)y|_{s=t}$.

LEMMA 1. *Let* $A\in D_s(\Omega,\mathscr{L}(\mathbb{F},\mathbb{G}))$, $B\in C(\Omega,\mathscr{L}(\mathbb{G},\mathbb{H}))$, *and* $C(t)=B(t)\circ A(t)$, $t\in\Omega$.

(a) *If* $B\in D_s(\Omega,\mathscr{L}(\mathbb{G},\mathbb{H}))$, *then* $C\in D_s(\Omega,\mathscr{L}(\mathbb{F},\mathbb{H}))$.

(b) *If* $C\in D_s(\Omega,\mathscr{L}(\mathbb{F},\mathbb{H}))$, *then for all* $t\in\Omega$, $y\in\mathbb{F}$, *the map* $s\to B(s)\circ A(t)y$ *possesses a Fréchet derivative at the point* $s=t$.

In both cases,

$$DC(t)y=DB(s)\circ A(t)y|_{s=t}+B(t)\circ(DA(t)y)$$

for all $t\in\Omega$ *and* $y\in\mathbb{F}$.

Proof. Let $s,t\in\Omega$, $y\in\mathbb{F}$,

$$\varphi(s)=[C(s)-C(t)]y-[B(s)-B(t)]\circ A(t)y-B(t)\circ(DA(t)y)(s-t),$$

$$\alpha(s)=B(s)\{[A(s)-A(t)]y-(DA(t)y)(s-t)\},$$

$$\beta(s)=[B(s)-B(t)]\circ(DA(t)y)(s-t).$$

Then $\varphi(s)=\alpha(s)+\beta(s)$. By the continuity of B and the definition of the derivative,

$$\lim_{\substack{s\to t\\ s\neq t}}\frac{1}{\|s-t\|}\alpha(s)=0.$$

On the other hand if $s\neq t$,

$$\frac{1}{\|s-t\|}\|\beta(s)\|\leqslant\|B(s)-B(t)\|\,\|DA(t)y\|;$$

hence, by the continuity of B,

$$\lim_{\substack{s\to t\\ s\neq t}}\frac{1}{\|s-t\|}\beta(s)=0,\qquad \lim_{\substack{s\to t\\ s\neq t}}\frac{1}{\|s-t\|}\varphi(s)=0.$$

(a): Assume that $B \in D_s(\Omega, \mathscr{L}(\mathbb{G},\mathbb{H}))$. Let

$$\gamma(s) = [C(s) - C(t)]y - DB(s) \circ A(t)y|_{s=t}(s-t)$$
$$- B(t) \circ (DA(t)y)(s-t).$$

Then

$$\gamma(s) = \varphi(s) + [B(s) - B(t)] \circ A(t)y - DB(s) \circ A(t)y|_{s=t}(s-t).$$

Hence, by the definition of the derivative,

$$\lim_{\substack{s \to t \\ s \neq t}} \frac{1}{\|s-t\|}\gamma(s) = 0$$

and the conclusion follows.

(b): Assume that $C \in D_s(\Omega, \mathscr{L}(\mathbb{F},\mathbb{H}))$. Let

$$\delta(s) = [B(s) - B(t)] \circ A(t)y - (DC(t)y)(s-t) + B(t) \circ (DA(t)y)(s-t).$$

Then

$$\delta(s) = -\varphi(s) + [C(s) - C(t)]y - (DC(t)y)(s-t),$$

and by the definition of the derivative

$$\lim_{\substack{s \to t \\ s \neq t}} \frac{1}{\|s-t\|}\delta(s) = 0.$$

Hence the conclusion follows. ■

Remark. In the particular case where $\Omega \subset \mathbb{K}^n = \mathbb{E}$ and $\mathbb{G}$ is complete, the hypothesis that B is continuous may be replaced by its being only strongly continuous in Lemma 1. The proof is similar, with the following modification: the strong continuity of B implies by the principle of uniform boundedness that

$$\sup_{s \in K} \|B(s)\| < \infty,$$

where K is any compact neighborhood of t.

LEMMA 2. *Let* $\psi \in C(\Omega, \mathscr{L}(\mathbb{F}))$ *and* $\pi \in D_s(\Omega, \mathscr{L}(\mathbb{G}, \mathbb{F}))$ *be such that at every point* $t \in \Omega$

$$\psi(t) \circ \pi(t) = \pi(t).$$

Moreover assume that at $t_0 \in \Omega$ *and* $z_0 \in \mathbb{G}$,

$$(D\pi(t_0)z_0)(x) \in \pi(t_0)(\mathbb{G})$$

for all $x \in \mathbb{E}$. *Then the map* $s \to \psi(s) \circ \pi(t_0)z_0$ *possesses a Fréchet derivative at the point* $s = t_0$, *and this derivative is null.*

Proof. By Lemma 1(b), the derivative

$$D\psi(s) \circ \pi(t_0)z_0|_{s=t_0}$$

exists and is equal to

$$D\pi(t_0)z_0 - \psi(t_0) \circ (D\pi(t_0)z_0).$$

Let $x \in \mathbb{E}$. By hypothesis, there is $z_1 \in \mathbb{G}$ such that

$$(D\pi(t_0)z_0)(x) = \pi(t_0)z_1.$$

Therefore

$$D\psi(s) \circ \pi(t_0)z_0|_{s=t_0}(x) = \pi(t_0)z_1 - \psi(t_0) \circ \pi(t_0)z_1 = 0. \qquad \blacksquare$$

REMARK. In addition to the remark above, in the particular case where $\Omega \subset \mathbb{K}^n = \mathbb{E}$ and $\mathbb{F}$ is complete, the hypothesis that ψ is continuous may be replaced by the hypothesis that it is only strongly continuous.

2. CONDITION FOR A VECTOR SUBSPACE $E(t)$ OF ANY DIMENSION TO BE CONSTANT

In the following theorem, we establish a condition for a vector subspace $E(t)$ to be locally constant. A condition for it to be globally constant should be established by using Lemma 2.1 of [2] as in Corollaries 4 and 5 below.

THEOREM 3. *Let $t_0 \in \Omega$, and let $E = (E(t))_{t \in \Omega}$ be a family of vector subspaces of $\mathbb{F}$ such that there is*

$$\pi \in D_s \cap C(\Omega, \mathscr{L}(\mathbb{E}_0, \mathbb{F})),$$

where $\mathbb{E}_0 = E(t_0)$, satisfying the conditions

$$E(t) = \pi(t)(\mathbb{E}_0),$$

$$(D\pi(t)z)(x) \in E(t)$$

for all $t \in \Omega$, $z \in \mathbb{E}_0$, $x \in \mathbb{E}$. Assume moreover that $\mathbb{E}_0$ is complete and there is $P \in \mathscr{L}(\mathbb{F}, \mathbb{E}_0)$ such that $P \circ \pi(t_0)$ is invertible in $\mathscr{L}(\mathbb{E}_0)$. Then E is constant on a neighborhood of t_0.

Proof. It follows from invertibility of $P \circ \pi(t_0)$, continuity of π, and completeness of $\mathbb{E}_0$ that $P \circ \pi(t)$ is invertible in $\mathscr{L}(\mathbb{E}_0)$ at every point t of a connected neighborhood $\Omega_0 \subset \Omega$ of t_0 (a sufficiently small ball of center t_0, for example). Let

$$\phi(t) = \pi(t) \circ [P \circ \pi(t)]^{-1},$$

$$\psi(t) = \phi(t) \circ P, \qquad t \in \Omega_0.$$

As the maps π, $A \mapsto A^{-1}$, $(A, B) \mapsto A \circ B$ are continuous, so are ϕ and ψ. Observing that

$$\psi(t) \circ \pi(t) = \pi(t), \qquad t \in \Omega_0,$$

and using the hypotheses, we may apply Lemma 2. We obtain that for all $z \in \mathbb{E}_0$, $t \in \Omega_0$, the map

$$s \mapsto \psi(s) \circ \pi(t)z = \phi(s) \circ P \circ \pi(t)z$$

is differentiable at the point $s = t$ and

$$D\phi(s) \circ P \circ \pi(t)z|_{s=t} = 0.$$

Let $t \in \Omega_0$. Since $(P \circ \pi(t))(\mathbb{E}_0) = \mathbb{E}_0$, it follows that $\phi \in D_s(\Omega_0, \mathscr{L}(\mathbb{E}_0, \mathbb{F}))$, and for all $z \in \mathbb{E}_0$, $D\phi(t)z = 0$. Therefore ϕ is constant, and for all $t \in \Omega_0$

$$E(t) = \phi(t)(\mathbb{E}_0) = \phi(t_0)(\mathbb{E}_0) = \mathbb{E}_0. \qquad \blacksquare$$

3. APPLICATION TO THE CASE WHERE $E(t)$ IS THE RANGE OF A PROJECTOR $P(t)$

COROLLARY 4. *Assume that Ω is connected and $\mathbb{F}$ is complete. Let $E = (E(t))_{t \in \Omega}$ be a family of vector subspaces of $\mathbb{F}$ such that there is*

$$P \in D_s \cap C(\Omega, \mathscr{L}(\mathbb{F}))$$

satisfying the conditions

$$E(t) = P(t)(\mathbb{F}),$$

$$P(t) \circ P(t) = P(t),$$

$$(DP(t)y)(x) \in E(t),$$

for all $t \in \Omega$, $y \in \mathbb{F}$, $x \in \mathbb{E}$. Then E is constant on Ω.

Proof. By Lemma 2.1 of [2], it is sufficient to prove that E is constant on a neighborhood of every point of Ω. Let $t_0 \in \Omega$, $\mathbb{E}_0 = E(t_0)$, $P_0 = P(t_0)$, and

$$\pi(t) = P(t)|_{\mathbb{E}_0}, \qquad t \in \Omega.$$

The hypotheses at once imply that

$$\pi \in D_s \cap C(\Omega, \mathscr{L}(\mathbb{E}_0, \mathbb{F}))$$

and

$$(D\pi(t)z)(x) \in E(t)$$

for all $t \in \Omega$, $z \in \mathbb{E}_0$, and $x \in \mathbb{E}$. Since P is continuous, by virtue of Lemma 3.3 of [2],

$$\pi(t)(\mathbb{E}_0) = E(t)$$

at every point t of a neighborhood $\Omega_0 \subset \Omega$ of t_0. As $\mathbb{F}$ is complete and

$$\mathbb{E}_0 = \operatorname{Ker}(P_0 - \mathrm{id}_{\mathbb{F}})$$

is closed, $\mathbb{E}_0$ itself is complete.

Lastly, $(\mathrm{id}_{\mathbb{E}_0} \circ P_0) \circ \pi(t_0) = \mathrm{id}_{\mathbb{E}_0}$ is invertible in $\mathscr{L}(\mathbb{E}_0)$, and $\mathrm{id}_{\mathbb{E}_0} \circ P_0 \in \mathscr{L}(\mathbb{F}, \mathbb{E}_0)$. Therefore, by Theorem 3, E is constant on a neighborhood of t_0. ■

4. APPLICATION TO THE CASE WHERE $E(t)$ IS FINITE-DIMENSIONAL

COROLLARY 5. *Assume that Ω is connected. Let $E = (E(t))_{t \in \Omega}$ be a family of vector subspaces of $\mathbb{F}$ of finite constant dimension m, such that there are $e_1, \dots, e_m \in D(\Omega, \mathbb{F})$ satisfying the conditions that at every point $t \in \Omega$, $(e_1(t), \dots, e_m(t))$ is a basis of $E(t)$, and for all $x \in \mathbb{E}$, $(De_1(t))(x), \dots, (De_m(t))(x) \in E(t)$. Then E is constant on Ω.*

Proof. As in the proof of Corollary 4, we may replace Ω by any open neighborhood of an arbitrary point $t_0 \in \Omega$. As $\mathbb{E}_0 = E(t_0)$ is finite-dimensional, it is complete, and by [1, Chapter II, Corollaries 2 of §4 and 3 of §8], there is $P \in \mathscr{L}(\mathbb{F}, \mathbb{E}_0)$ such that $P|_{\mathbb{E}_0} = \mathrm{id}_{\mathbb{E}_0}$.

Define $\pi : \Omega \to \mathscr{L}(\mathbb{E}_0, \mathbb{F})$ by

$$\pi(t)\left(\sum_{i=1}^{m} x_i e_i(t_0)\right) = \sum_{i=1}^{m} x_i e_i(t),$$

for all $t \in \Omega$, $x_1, \dots, x_m \in \mathbb{K}$. It is easy to verify that the hypotheses of Theorem 3 are satisfied and therefore E is constant on a neighborhood of t_0. ■

5. CONDITION FOR A PROJECTOR $P(t)$ TO BE CONSTANT

If a projector $P(t)$ is constant, so obviously is its range $E(t) = P(t)(\mathbb{F})$, but the converse is false. For example the projector

$$P(t) = \frac{1}{2}\begin{pmatrix} 1-t & 1+t \\ 1-t & 1+t \end{pmatrix}, \qquad t \in \mathbb{R},$$

is not constant, although its range

$$E(t) = P(t)(\mathbb{R}^2) = \mathbb{R}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad t \in \mathbb{R},$$

is. Obviously its direction of projection

$$d(t) = \begin{pmatrix} t+1 \\ t-1 \end{pmatrix}, \qquad t \in \mathbb{R},$$

is not constant.

If $P(t)$ commutes with its derivative, its range $E(t)$ is invariant under this derivative and, by Corollary 4, $E(t)$ is constant. We will see in two ways that this hypothesis implies that not only is $E(t)$ constant, but also $P(t)$ is.

THEOREM 6. *Assume that Ω is connected. Let*

$$P \in D_s \cap C(\Omega, \mathscr{L}(\mathbb{F}))$$

be a projector function commuting with its derivative, that is to say,

$$P^2(t) = P(t),$$

$$DP(s) \circ P(t) y|_{s=t} = P(t) \circ (DP(t) y)$$

for all $t \in \Omega$, $y \in \mathbb{F}$. Then P is constant on Ω.

Proof. Let $t \in \Omega$ and $y \in \mathbb{F}$. By Lemma 1(a),

$$DP(t) y = DP^2(t) y = DP(s) \circ P(t) y|_{s=t} + P(t) \circ (DP(t) y).$$

Since P commutes with its derivative,

$$DP(t) y = P(t) \circ (2DP(t) y).$$

Hence for all $x \in \mathbb{E}$

$$(DP(t) y)(x) \in P(t)(\mathbb{F}).$$

By applying Lemma 2 with $\mathbb{G} = \mathbb{F}$ and $\psi = \pi = P$, we obtain

$$DP(s) \circ P(t)y|_{s=t} = 0,$$

and by using a second time the commutation of P with its derivative, we get

$$DP(t)y = 2DP(s) \circ P(t)y|_{s=t} = 0.$$

Hence the conclusion follows, because Ω is connected. ∎

REMARKS.

(a) Theorem 6 also may be proved algebraically in the following way:

$$DP = 2P(DP)$$

as above. Hence, by multiplying this equation on the left by P,

$$P(DP) = 2P^2(DP) = 2P(DP),$$

which implies

$$P(DP) = 0$$

and

$$DP = 2P(DP) = 0.$$

(b) In the same particular case as in the remark after Lemma 2, the hypothesis that P is continuous may be suppressed in the statement of Theorem 6.

The following corollary is an obvious consequence of Theorem 6.

COROLLARY 7. *Assume that Ω is connected. Let $P \in D(\Omega, \mathscr{L}(\mathbb{F}))$ be a projector function commuting with its derivative. Then P is constant on Ω.*

APPLICATION. The proof of Theorem 3.5 of [2] is a typical example where this corollary is needed.

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